100 Years
NORTH SYDNEY BOYS HIGH SCHOOL
1912 ~ 2012

# MATHEMATICS (EXTENSION 1)

2013 HSC Course Assessment Task 4
Friday August 16, 2013

**General instructions**

- Working time – 50 min.
  (plus 5 minutes reading time)
- **Commence each new question on a new page**.
- Write using blue or black pen. Where diagrams are to be sketched, these may be done in pencil.
- Board approved calculators may be used.
- All necessary working should be shown in every question. Marks may be deducted for illegible or incomplete working.
- Attempt **all** questions.
- At the conclusion of the examination, bundle the booklets used in the correct order within this paper and hand to examination supervisors.

**Class** (please ✔)

- ◯ 12M3A – Mr Lam
- ◯ 12M3B – Mr Berry
- ◯ 12M3C – Mr Lin
- ◯ 12M4A – Mr Choy
- ◯ 12M4B – Mr Weiss
- ◯ 12M4C – Ms Ziaziaris

**STUDENT NUMBER** .................... **# BOOKLETS USED**: .................

Marker's use only.

| **QUESTION** | 1 | 2 | 3 | 4 (ab) | 4 (cd) | **Total** | % |
|---|---|---|---|---|---|---|---|
| **MARKS** | $\overline{4}$ | $\overline{8}$ | $\overline{6}$ | $\overline{4}$ | $\overline{5}$ | $\overline{27}$ | |

**Question 1** (4 Marks) Commence a NEW page. **Marks**

In the expansion of $\left(\dfrac{2}{x} - \dfrac{x^2}{3}\right)^{12}$

(a) Write the expansion in sigma notation. **2**

(b) Find the term independent of $x$ in the expansion. **2**

**Question 2** (8 Marks) Commence a NEW page. **Marks**

Mr Cash is borrowing \$500 000 to purchase a property. The bank will be lending him the amount based on a 6% p.a. interest rate on the balance owing, and loan term of 30 years with equal monthly repayments of $\$M$. $\$A_n$ is the amount owing after the $n$-th monthly repayment.

(a) Show that $A_n = 500\,000 \times 1.005^n - 200M\left(1.005^n - 1\right)$. **3**

(b) Hence or otherwise, show that the repayment amount $M = \$2997.75$. **1**

(c) Show that, if Mr Cash decides to make repayments of \$3 200 on a monthly basis, that he would be able to save approximately 55 months off the term of the loan. **2**

(d) Hence or otherwise, find the amount of interest saved by making repayments of \$3 200 instead of \$2 997.75 per month. **2**

(For simplicity, you may assume the final repayment for the additional repayment schedule is also \$3 200).

**Question 3** (6 Marks) Commence a NEW page. **Marks**

A particle is moving on a horizontal line with velocity $v$ ms$^{-1}$, with $v^2$ given as

$$v^2 = 64 + 24x - 4x^2$$

(a) Prove that the particle is moving in simple harmonic motion. **2**

(b) Find the centre of the motion. **1**

(c) Find the period and amplitude. **3**

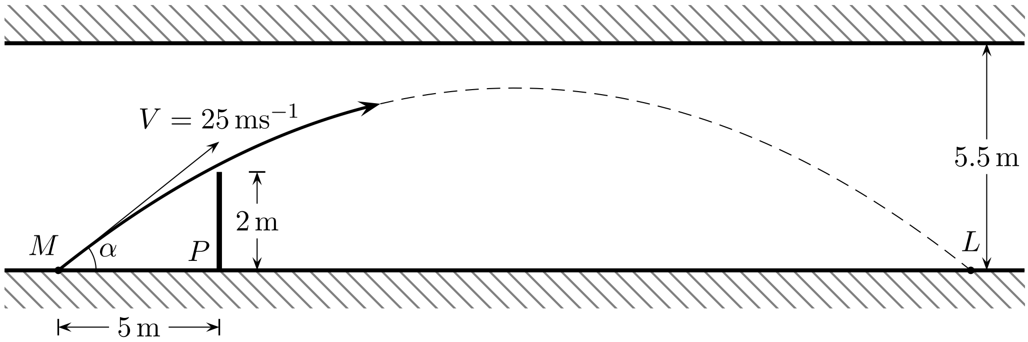
**Question 4** (9 Marks) Commence a NEW page. **Marks**

A movie scene is to be filmed inside a tunnel of height 5.5 m. In this scene, a stunt man (of negligible height) will ride a motorcycle off an inclined plane (in the shape of a wedge, of negligible height) positioned at point $M$ at $\alpha$ degrees, with velocity $V = 25\,\text{ms}^{-1}$.

The stunt man must just clear a policeman of height 2 m, standing 5 m away from the inclined plane at point $P$.

The stunt man will then land safely at another point $L$ in the tunnel.

Assume that there is no air resistance in the tunnel, and that the acceleration due to gravity is $g = 10\,\text{ms}^{-2}$.




You may also assume the equations of motion are $\begin{cases} x = 25t\cos\alpha \\ y = 25t\sin\alpha - 5t^2 \end{cases}$

and the equations for the velocity are $\begin{cases} \dot{x} = 25\cos\alpha \\ \dot{y} = 25\sin\alpha - 10t \end{cases}$.

(a) Show that the trajectory can be expressed as **2**

$$y = -\frac{x^2}{125}\sec^2\alpha + x\tan\alpha$$

(b) Hence show that $\tan^2\alpha - 25\tan\alpha + 11 = 0$. **2**

(c) Hence or otherwise, show that the value(s) of $\alpha$ (correct to the nearest degree) are **2**

$$\alpha = 24^\circ \text{ or } 88^\circ$$

(d) Explain (with mathematical reasoning) which of these values of $\alpha$ is valid. **3**

For simplicity, the rounded off values from part (c) may be used in your explanation

**End of paper.**

## STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1} + C, \qquad n \neq -1; \quad x \neq 0 \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x + C, \qquad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax} + C, \qquad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax + C, \qquad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax + C, \qquad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax + C, \qquad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax + C, \qquad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a} + C, \qquad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a} + C, \qquad a > 0, -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right) + C, \qquad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right) + C$$

NOTE: $\ln x = \log_e x, \; x > 0$

## Suggested Solutions

**Question 1** (Ziaziaris)

(a) (2 marks)

$$\begin{aligned}&\left(\frac{2}{x}-\frac{x^2}{3}\right)^{12}\\&=\sum_{k=0}^{12}\binom{12}{k}2^k\left(x^{-1}\right)^k(-1)^{12-k}\left(3^{-1}\right)^{12-k}\left(x^2\right)^{12-k}\\&=\sum_{k=0}^{12}\binom{12}{k}2^k(-1)^{12-k}3^{k-12}x^{-k+24-2k}\\&=\sum_{k=0}^{12}\binom{12}{k}2^k(-1)^{12-k}3^{k-12}x^{24-3k}\end{aligned}$$

(b) (2 marks) Term independent of $x$ appears when

$$\begin{aligned}24-3k&=0\\3k&=24\\k&=8\end{aligned}$$

Hence, the constant term is

$$\binom{12}{8}2^8 3^{-4}$$

**Question 2** (Lin)

(a) (3 marks)

- At the end of the first month: $r=\frac{0.06}{12}=0.005$ p.m.

$$A_1=500\,000\times 1.005-M$$

- At the end of the second month:

$$\begin{aligned}A_2&=A_1\times 1.005-M\\&=(500\,000\times 1.005-M)\times 1.005-M\\&=500\,000\times 1.005^2-M(1+1.005)\end{aligned}$$

- At the end of the third month:

$$\begin{aligned}A_3&=A_2\times 1.005-M\\&=500\,000\times 1.005^3\\&\quad-M(1+1.005+1.005^2)\end{aligned}$$

- At the end of $n$ months,

$$\begin{aligned}A_n&=500\,000\times 1.005^n\\&-M\underbrace{\left(1+1.005+1.005^2+\cdots+1.005^{n-1}\right)}_{S_n:\ r=1.005,\ a=1,\ n\text{ terms}}\end{aligned}$$

$$\begin{aligned}S_n&=\frac{a(r^n-1)}{r-1}\\&=\frac{1.005^n-1}{1.005-1}\\&=200\left(1.005^n-1\right)\end{aligned}$$

$$\therefore A_n=500\,000\times 1.005^n-200M\left(1.005^n-1\right)$$

(b) (1 mark)
At the end of the loan of 30 years, $n=360$, $A_{360}=0$:

$$\begin{aligned}500\,000\times 1.005^{360}-200M\left(1.005^{360}-1\right)&=0\\200M\left(1.005^{360}-1\right)&=500\,000\times 1.005^{360}\\\therefore M=\frac{500\,000\times 1.005^{360}}{200\left(1.005^{360}-1\right)}&=\$2997.75\end{aligned}$$

(c) (2 marks) If $M=\$3\,200$,

$$A_n=500\,000\times 1.005^n-200(3\,200)\left(1.005^n-1\right)$$

At the end of the loan, $A_n=0$:

$$\begin{aligned}500\,000\times 1.005^n-200(3\,200)\left(1.005^n-1\right)&=0\\1.005^n\left(500\,000-640\,000\right)+640\,000&=0\\\therefore 1.005^n\times 140\,000&=640\,000\\1.005^n&=\frac{64}{14}\\n\log 1.005&=\log\frac{64}{14}\\n=\frac{\log\frac{32}{7}}{\log 1.005}&=304.72\approx 305\text{ months}\end{aligned}$$

Previous repayment schedule had 360 months. Increasing repayments to \$3 200 per month now only takes 305 months. Hence a time saving of 55 months.

(d) (2 marks)

- \$2 997.75 per month repayments:
  - Total repaid:

    $$\$2997.75\times 360=\$1\,079\,190$$

  - Total interest:

    $$\$1\,079\,190-\$500\,000=\$579\,190$$

- \$3 200 per month repayments:
  - Total repaid:

    $$\$3200 \times 304.72 = \$975\,104$$

    Students may use 305 months: \$976 000
  - Total interest:

    $$\$975\,104 - \$500\,000 = \$475\,104$$

    (\$476 000)
- Total interest saved over 55 months:

  $$\$579\,190 - \$475\,104 = \$104\,086$$

  (\$103 190 if 305 mths used)

**Question 3** (Berry)

(a) (2 marks)

$$\begin{aligned}\ddot{x} &= \frac{d}{dx}\left(\frac{1}{2}v^2\right)\\ &= \frac{d}{dx}\left(\frac{1}{2}\left(64+24x-4x^2\right)\right)\\ &= \frac{d}{dx}\left(32+12x-2x^2\right)\\ &= 12-4x\\ &= -4(x-3)\\ &\equiv -n^2(x-k)\end{aligned}$$

As acceleration is proportional to, and directed against the motion, the particle is moving in simple harmonic motion.

(b) (1 mark) Centre: $x = 3$.

(c) (3 marks)

$$\begin{aligned}v^2 &= 64+24x-4x^2\\ &= -4\left(x^2-6x-16\right)\\ &= -4\left(x^2-6x+9-25\right)\\ &= -4\left((x-3)^2-25\right)\\ &= 4\left(25-(x-3)^2\right)\\ &\equiv n^2\left(a^2-(x-k)^2\right)\\ T &= \frac{2\pi}{n} = \frac{2\pi}{2} = \pi\\ a &= 5\end{aligned}$$

**Question 4** (Weiss (a)(b), Choy (c)(d))

(a) (2 marks)

$$\begin{cases} x = 25t\cos\alpha & (1)\\ y = 25t\sin\alpha - 5t^2 & (2)\end{cases}$$

Change subject of (1) to $t$, and substitute to (2):

$$\begin{aligned}t &= \frac{x}{25\cos\alpha}\\ y &= \cancel{25}\left(\frac{x\sin\alpha}{\cancel{25}\cos\alpha}\right) - 5\left(\frac{x}{25\cos\alpha}\right)^2\\ &= x\tan\alpha - \frac{5x^2}{625\cos^2\alpha}\\ &= x\tan\alpha - \frac{x^2}{125}\sec^2\alpha\end{aligned}$$

(b) (2 marks) When $x = 5$, $y = 2$:

$$\begin{aligned}2 &= 5\tan\alpha - \frac{5^2}{125}\left(1+\tan^2\alpha\right)\\ \underset{\times 5}{2} &= \underbrace{5\tan\alpha - \frac{1}{5}\left(1+\tan^2\alpha\right)}_{\times 5}\\ 10 &= 25\tan\alpha - 1 - \tan^2\alpha\\ \therefore \tan^2\alpha &- 25\tan\alpha + 11 = 0\end{aligned}$$

(c) (2 marks) Let $z = \tan\alpha$.

$$\begin{aligned}z^2 - 25z + 11 &= 0\\ z &= \frac{25 \pm \sqrt{25^2 - 4(1)(11)}}{2}\\ &= \frac{25\pm\sqrt{581}}{2}\\ &= \frac{25 \pm 24.1\cdots}{2}\\ \tan\alpha &= 24.5519 \text{ or } 0.4480\\ \therefore \alpha &= 87.6676^\circ \text{ or } 24.1337^\circ\\ &\approx 88^\circ \text{ or } 24^\circ\end{aligned}$$

(d) (3 marks)

- If $\alpha = 88^\circ$, check maximum height – i.e $\dot{y} = 0$

$$\begin{aligned}\dot{y} = 0 &= 25\sin 88^\circ - 10t\\ \therefore 10t &= 25\sin 88^\circ\\ t &= \frac{25\sin 88^\circ}{10} \approx 2.498\,\text{s}\end{aligned}$$

Check maximum height attainable with this angle of inclination:

$$\begin{aligned} y &= 25t\sin 88^\circ - 5t^2 \\ &= 25(2.498)\sin 88^\circ - 5(2.498)^2 \\ &\approx 31.2\,\text{m} \end{aligned}$$

i.e. if the inclined plane is at $88^\circ$, the stunt man will hit the roof well before he reaches his maximum height. Therefore an unsafe landing.

- If $\alpha = 24^\circ$, check maximum height – i.e $\dot{y} = 0$

$$\begin{aligned} \dot{y} = 0 &= 25\sin 24^\circ - 10t \\ \therefore 10t &= 25\sin 24^\circ \\ t &= \frac{25\sin 24^\circ}{10} \approx 1.016\,8\,\text{s} \end{aligned}$$

Check maximum height attainable with this angle of inclination:

$$\begin{aligned} y &= 25t\sin 24^\circ - 5t^2 \\ &= 25(1.0168)\sin 24^\circ - 5(1.0168)^2 \\ &\approx 5.169\,8\,\text{m} \end{aligned}$$

The maximum height attained by the stunt man when $\alpha = 24^\circ$ will be approx 5.2 m, well below the 5.5 m roof. Therefore he will land safely inside the tunnel.